Mathematical Logic and Computability

Lecture 1: Syntax of First-Order Logic

Yunpyo An

Ulsan National Institute of Science and Technology

Mathematical Logic and Computability
Sep 1, 2023

Why we study Mathematical Logic?

In this course, we will study mathematical logic and computability theory. What is the *logic*? Why we need it for us?
Logic is the study of the principles of correct reasoning. Logic consider formal or informal language and prove that with induction or model-theoretic method.
Among logic, mathematical logic is to find reasoning in mathematics (with mathematical reasoning). We will study and explore the *first-order logic* for expressing mathematics formally.

Why we study Computability?

Let's think about another simple question: What is computable? We can say about that "We can compute 1+1=2, 2+2=4, 3+3=6, ...". But, how can we say about that?
How can we construct a machine that can compute 1+1=2, 2+2=4, 3+3=6, ...? Can we compute all of the functions that we can imagine? We already know about the answer of this question is '**no**' by the Halting Problem. But we need to consider foundation of this answer, computation model.

Outline of Course

In this course, we will study following topics in mathematical logic:

- Syntax of First-Order Logic
- Semantics of First-Order Logic
- Sequent Calculus
- Completeness Theorem
- The Löwenheim-Skolem Theorem and Compactness Theorem

Outline of Course

In this course, we will study following topics in computability theory:

- Introduction to Computable Functions by Unlimited Register Machine
- Build on Computable Functions
- Turing Machines and Church-Turing Thesis
- Numbering of Computable Functions
- s-m-n Theorem, Universal Programs
- Deciability and Undecidability (Halting Problem)
- Gödel's Incompleteness Theorem

Course in Catalog

Before this course, you have taken following courses:

- Discrete Mathematics
- Programming Languages
- Introduction to Algorithms

After this course, you will discuss following topics:

- Type, Proof, Model Theory
- Set Theory
- Computability Theory
- (further) Computational Complexity Theory

Textbook

In this course, in mathematical logic, we will use following textbook (In my opinion, this is not friendly and introductory textbook):

- *Mathematical Logic*, by H.-D. Ebbinghaus, J. Flum, and W. Thomas

In this course, in computability theory, we will use following textbook:

- *Computability: An Introduction to Recursive Function Theory*, by N. J. Cutland

Textbook

In this course, I recommend following textbook:

- *Introduction to Mathematical Logic*, by E. Mendelson
- *Mathematical Logic*, by J. Shoenfield
- *Computability and Logic*, by G. S. Boolos, J. P. Burgess, and R. C. Jeffrey
- *Computability and Unsolvability*, by M. Davis

Also you can use following textbook:

- *Beginning Mathematical Logic: A Study Guide*, by Peter Smith Online
- *Open Logic Project*, by Richard Zach and others Online

History of Logic: Acient Logic

우리는 논리학의 출발지점을 철학사상의 출발 지점인 고대 그리스에서 살펴볼 수 있다. 인간의 논리적 사유는 인류의 시작과 더불어 생겨났음이 분명하나, 이를 기록하여 체계적인 학문으로 성립한 것은 아리스토텔레스의 *오르가논* 이다. 그 이전의 논리학의 경우는 아래와 같은 문제를 생각해볼 수 있다.

- 기하학에서 추상적인 명제의 참을 증명하려는 시도
- 형이상학에서 세계의 구조에 관한 일반적인 명제를 증명하려는 시도
- 일반적인 논증에서 우연적인 명제를 증명하고 논박하는 과정

별도의 학문으로써 자리잡지는 않았지만 이러한 시도들을 피타고라스, 유클리드, 플라톤 등에서 살펴볼 수 있다. 고대 논리학의 경우에는 상대의 논증을 논박하는 과정에서 논증의 구조를 분석하고, 이를 기반으로 논증의 구조를 분류하였다.

History of Logic: Boolean Logic

논리학을 계산 체계로써 제시하고, 수학에 편입하려는 시도는 Leibniz에서 시작하였다. 추상적인 형식성을 제시하고 이를 기반으로 논증의 구조를 분석하려는 시도는 George Boole 의 논리학에서 시작하였다.
Boole 이전의 논리학은 자유학예로써 자유인으로써 필요한 교양과정으로 인식이 되었다. 여기서의 논리학은 변증법과 같은 설득력 있는 이야기를 하기 위한 수단으로써 사용되었다. 아마도 산수와 논리학은 서로 다른 분야로써 탐구가 되었다. George Boole의 업적은 논리학을 대수적인 방법으로 설명하였다는 점이다. 지금 우리가 활용하는 진리함수나 선언 표준형 등이 Boole의 논리학에서 시작되었다. 이는 논리학을 수학의 일부로써 편입시키는 데에 큰 역할을 하였다.

History of Logic: Hilbert's Program

Boole 이후 논리학은 Frege의 논리주의, Cantor의 집합론, Peano의 산술 등의 발전을 거쳐 현대 논리학의 기반이 되었다. Hilbert는 이러한 발전에 힘입어 논리학을 수학에 편입시킨거와 마찬가지로 수학을 논리학을 통해서 엄밀하게 기술하고자 하였다. 이를 위해서 Hilbert는 논리학의 기초를 수학의 기초로써 제시하고자 하였다. 이를 Hilbert's Program 이라고 한다.
이러한 Hilbert의 꿈은 Gödel의 불완전성 정리에 의해서 무너지게 되었다. Gödel의 불완전성 정리는 논리학의 기초를 수학의 기초로써 제시하는 것이 불가능하다는 것을 보여주었다. 자연수 체계 안에서 우리는 참 혹은 거짓을 모두 도출하는 명제를 만들 수 있고 (모순성), 이는 Continuum Hypothesis와 같은 문제를 만들어낸다.

History of Logic

If you are interest in the history of logic, you can read the following books:

- *The Development of Logic* by William and Martha Kneale (I refer this book to write this section)
- *A Bibliography of Symbolic Logic* by Alonzo Church, Journal of Symbolic Logic
- *Handbook of History of Logic* edited by Dov Gabbay and John Woods

Key Concepts in Our Journey

Before starting our journey, we will introduce key concepts in our journey. It's hard to look at the forest why you study these contents from the beginning.

Syntax and Semantics of First-order Logic

Syntax is a study of the structure of sentences. Semantics is a study of the meaning of sentences.
Huh? What is the difference between syntax and semantics? Thinking about this sentence "ϕ is a student of UNIST". (ϕ has no meaning.) We can say that this sentence is syntactically (or grammaly) correct but semantically incorrect. How about assign ϕ into Yunpyo? Then, this sentence is semantically correct.

Example of First-order Logic: Groups

Definition

A *group* is a triple $(G, \circ^G, e^G)$ which satisfy followings:

- $\forall x \forall y \forall z \left((x \circ^G y) \circ^G z = x \circ^G (y \circ^G z)\right)$
- $\forall x \left(x \circ^G e^G = x\right)$
- $\forall x \exists y \left(x \circ^G y = e^G\right)$

where G is a set, $\circ^G$ is a binary operation on G, and e^G is a constant symbol.

Question: Make a formal sentence "for every x there exists y such that $y \circ^G x = e^G$". And show that is a consequence of the axioms of group. How did you prove this?

Sequent Calculus and Proof

In previous question, we prove that from the set of sentence to another sentence. We don't care about the meaning of the sentence. We write the sentence from the set of sentence to another sentence. We just care about the structure of the sentence. This is the main idea of sequent calculus.

Definition

A *sequent* is a pair of sets of formulas Γ and Δ. We write $\Gamma \vdash \Delta$ to denote a sequent.

Consequence and Model of First-order Logic

So, what is the semantically true conclusion (or consequence)? If there is an interpretation that makes all the preceding sentences true, the conclusion must be true to be correct reasoning.

Definition

A *model* of a set of sentences Γ is an interpretation $\mathcal{I}$ assign to each variable, constant, function, and relation symbol in Γ.

Definition

A *consequence* of a set of sentences Γ is a sentence ϕ such that every model of Γ is also a model of ϕ. ($\Gamma \models \phi$)

Consistency of First-order Logic

Definition

A set of sentences Γ is *consistent* if there is no sentence ϕ such that $\Gamma \vdash \phi$ and $\Gamma \vdash \neg\phi$. (Consistency)

Completeness of First-order Logic

Okay... Question: If ϕ is a consequence of Γ, then $\Gamma \vdash \phi$? or $\Gamma \vdash \phi$ then ϕ is a consequence of Γ? Can we prove this?

Definition

For all Γ and ϕ If $\Gamma \models \phi$, then $\Gamma \vdash \phi$. (Completeness)

Completeness of First-order Logic

The answer of previous question is **yes** in first-order logic.

Gödel's Completeness Theorem

In first-order logic, for all Γ and ϕ If $\Gamma \models \phi$, then $\Gamma \vdash \phi$. (Completeness)

In other words, if ϕ is a consequence of Γ, then $\Gamma \vdash \phi$.

Completeness of First-order Logic

But the natural number system cannot be axiomatized in first-order logic. Also, in higher-order logic, the answer is **no**. (Beyond our scope. If you are interest in this topic, you can read further part of Ebbinghaus textbook)

Gödel's Incompleteness Theorem

First, any consistent formal system F within which a certain amount of elementary arithmetic can be carried out is incomplete.
In other words, there are statements of the language of F which can neither be proved nor disproved in F. (Raatikainen 2020)
Second, for any consistent system F within which a certain amount of elementary arithmetic can be carried out, the consistency of F cannot be proved in F itself. (Raatikainen 2020)

Löwenheim-Skolem Theorem

Löwenheim-Skolem Theorem

Every at most countable and satisfiable set of formulas is satisfiable over a domain which is at most countable.
i.e., it has a model whose domain is at most countable.

Limitation of First-order Logic

Peano's axioms

Peano suggested the following axioms for the natural numbers:

- 0 is not a successor of any natural number.
- successor is injective.
- For any subset of X of $\mathbb{N}$ if $0 \in X$ and if $\sigma(n) \in X$ whenever $n \in X$, then $X = \mathbb{N}$. (*induction axiom*)

Note that, the third axiom (*induction axiom*) is not a first-order logic sentence. (Can we express it with only one quantifier over set?) There are try to express natural number system with first-order logic (as Presburger arithmetic), but it is not possible.