# Mathematical Logic and Computability

## Lecture 3: Semantics of First-Order Logic

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# Review

In previous lecture, we learned the syntax of first-order logic.

- Alphabet of First-order logic
- Terms
- Formulas
- Free and bound variables
- Substitution

Now, we assign the Mathematical object to each symbol in first-order language.

# Structures

Before define the semantics of first-order logic, we need to consider our language domain.

## Definition

An $S$-structure is a pair $\mathcal{A} = (A, \mathfrak{a})$ with the following properties:

- $A$ is a non-empty set, called the *domain* or *universe* of $\mathcal{I}$
- $\mathfrak{a}$ is a function that assigns from symbols to following:
  - for every $n$-ary relation symbol $R$ in $S$, $\mathfrak{a}(R)$ is an $n$-ary relation on $A$
  - for every $n$-ary function symbol $f$ in $S$, $\mathfrak{a}(f)$ is an $n$-ary function on $A$
  - for every constant $c$ in $S$, $\mathfrak{a}(c)$ is an element of $A$

From Ebbinghaus textbook, for convenience, we denote $\mathfrak{a}(R), \mathfrak{a}(f), \mathfrak{a}(c)$ by $R^{\mathcal{A}}, f^{\mathcal{A}}, c^{\mathcal{A}}$ or $R^A, f^A, c^A$respectively.

# Example of Structures

We write $S = \{R, f\}$, where $R$ is a n-ary relation symbol, $f$ is a n-ary function symbol. The structure of $S$ is denote as $\mathcal{A} = (A, R^{\mathcal{A}}, f^{\mathcal{A}})$.
We consider symbol set of arithmetic as follows:

$$S_{\mathrm{ar}} := \{+, \cdot, 0, 1\} \quad \text{and} \quad S_{\mathrm{ar}}^{<} := \{+, \cdot, 0, 1, <\} \tag{1}$$

We will use $\mathcal{N}$ as the structure of natural number arithmetic with $S_{\mathrm{ar}}$ (equation 1).

$$\mathcal{N} := (\mathbb{N}, +^{\mathbb{N}}, \cdot^{\mathbb{N}}, 0^{\mathbb{N}}, 1^{\mathbb{N}})$$

where, our domain is $\mathbb{N}$, $+$ and $\cdot$ are addition and multiplication, 0 and 1 are zero and one respectively.

# Structures and Interpretations

We remain the variable symbols for semantics of first-order logic. We **assign** a value in our domain $A$ to each variable.

## Definition

An *assignment* in $S$-structure $\mathcal{A}$ is a function $\beta : \{v_n | n \in \mathbb{N}\} \to A$ from the set of variables into the domain $A$.

# Structures and Interpretations

Now, we combine structure and interpretations together.

## Definition

An $S$-interpretation $\mathcal{I}$ is a pair $(\mathcal{A}, \beta)$, where $\mathcal{A}$ is an $S$-structure and $\beta$ is an assignment in $\mathcal{A}$.

# Structures and Interpretations

We might consider assignment is a subtitution of variables to values in domain. We can write as follows:

$$\beta\frac{a}{x}(y) := \begin{cases} \beta(y) & \text{otherwise} \\ a & \text{if } y = x \end{cases}$$

# Interpretations

Let's define interpretation $\mathcal{I} = (\mathcal{A}, I)$ is given by

$$\mathcal{I} = (\mathbb{N}, +, \cdot, 0, 1, <) \quad \text{and} \quad \beta(v_n) = 2n \text{ for } n \geq 0$$

## Example

The formula $v_2 \cdot (v_1 + v_2) \equiv v_4$ reads as $4 \cdot (2 + 4) \equiv 8$.

**Question**: Interprete the follwing formulas by $\mathcal{I}$.

$$\exists v_0 v_0 + v_0 \equiv v_1 \tag{2}$$

$$\forall v_0 \forall v_1 \exists v_2 (v_0 < v_2 \wedge v_2 < v_1) \tag{3}$$

# Connectives

As we learned in propositional logic, we need to define the semantics of connectives with truth-table.

| | | $\dot{\vee}$ | $\dot{\wedge}$ | $\dot{\rightarrow}$ | $\dot{\leftrightarrow}$ |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | F | T | F | F | F |
| F | T | T | F | T | F |
| F | F | F | F | T | T |

| | $\dot{\neg}$ |
|---|---|
| T | F |
| F | T |

# The Satisfication Relation

Now, we define interprete of our $S$-formula. Let's given $S$-formula $\varphi$ and $S$-interpretation $\mathcal{I} = (\mathcal{A}, \beta)$. Interpreted result is denoted by $\mathcal{I}(\varphi)$. We define $\mathcal{I}(\varphi)$ by induction on terms

## Definition

- For a variable $x$ let $\mathcal{I}(x) = \beta(x)$
- For a constant $c \in S$ let $\mathcal{I}(c) = c^{\mathcal{A}}$
- For $n$-ary function symbol $f \in S$ and terms $t_1, \ldots, t_n$ let $\mathcal{I}(f(t_1, \ldots, t_n)) = f^{\mathcal{A}}(\mathcal{I}(t_1), \cdots, \mathcal{I}(t_n))$

# The Satisfication Relation

For all interpretations $\mathcal{I} = (\mathcal{A}, \beta)$ we define following interpretations

- $\mathcal{I} \models (t_1 \equiv t_2)$ iff. $\mathcal{I}(t_1) = \mathcal{I}(t_2)$
- $\mathcal{I} \models (Rt_1 \dots t_n)$ iff. $R^{\mathcal{A}}(\mathcal{I}(t_1), \dots, \mathcal{I}(t_n))$
- $\mathcal{I} \models (\neg\varphi)$ iff. not $\mathcal{I} \models \varphi$
- $\mathcal{I} \models (\varphi \wedge \psi)$ iff. $\mathcal{I} \models \varphi$ and $\mathcal{I} \models \psi$
- $\mathcal{I} \models (\varphi \vee \psi)$ iff. $\mathcal{I} \models \varphi$ or $\mathcal{I} \models \psi$
- $\mathcal{I} \models (\varphi \rightarrow \psi)$ iff. $\mathcal{I} \models \varphi$ implies $\mathcal{I} \models \psi$
- $\mathcal{I} \models (\varphi \leftrightarrow \psi)$ iff. $\mathcal{I} \models \varphi$ iff. $\mathcal{I} \models \psi$
- $\mathcal{I} \models (\forall x\varphi)$ iff. for all $a \in A$, $\mathcal{I}\frac{a}{x} \models \varphi$
- $\mathcal{I} \models (\exists x\varphi)$ iff. there exists $a \in A$, $\mathcal{I}\frac{a}{x} \models \varphi$

# The Consequence Relation

### Definition
Let $\Phi$ be a set of $S$-formulas and $\varphi$ be an $S$-formula. We say that *$\phi$ is a consequence of $\Phi$ (written $\Phi \models \varphi$)* iff. for every $S$-interpretation $\mathcal{I}$ if $\mathcal{I} \models \psi$ for all $\psi \in \Phi$, then $\mathcal{I} \models \varphi$.

### Definition
A formula $\varphi$ is valid (written $\models \varphi$) iff. $\emptyset \models \varphi$.

### Definition
A formula $\varphi$ is *satisfiable* (written $Sat\, \varphi$) if and only if there is interpretation which is a model of $\varphi$. A set of formula $\Phi$ is *satisfiable* if and only if there is interpretation which is a model of $\Phi$.

Note. The satisfiability of formula is called SAT problem. It is one of the most important problem in computer science and its complexity is NP-hard.

# Example of Consequence Relation



In previous lecture, we consider about left inverse of group. Let's the set of axiom of group formula as $\Phi_{\mathrm{gr}}$.
Now we can formulate the left inverse of group as follows:

$$\Phi_{\mathrm{gr}} \models \{\forall v_0 \exists v_1 (v_1 \cdot v_0 \equiv e)\} \tag{4}$$

where the axiom of group as follows:

$$\begin{aligned}\Phi_{\mathrm{gr}} = \{&\forall v_0 \forall v_1 \forall v_2 (v_0 \cdot (v_1 \cdot v_2) \equiv (v_0 \cdot v_1) \cdot v_2), \\ &\forall v_0 (v_0 \cdot e \equiv v_0), \forall v_0 \exists v_1 (v_1 \cdot v_0 \equiv e)\}\end{aligned}$$

# Satisfiability and Validity



### Lemma

*For all $\Phi$ and $\varphi$,*

$$\Phi \models \varphi \quad \textit{iff.} \quad \textit{not } Sat \Phi \cup \{\neg\varphi\}$$

**Question**: Prove it.

*Hint*: Use the definition of consequence relation.

# Coincidence

The satisfication relation between $S$-formula $\varphi$ and an $S$-interpretation $\mathcal{I}$ depends only on interpretation of the symols of $S$ occuring in $\varphi$, and on the variable occurring free in $\varphi$.

## Definition

Two interpretation $\mathcal{I}_1$ and $\mathcal{I}_2$ agree on $k \in S$ on $x$ if $k^{\mathcal{A}_1} = k^{\mathcal{A}_2}$ or $\beta_1(x) = \beta_2(x)$.

## Lemma

*Let's $\mathcal{I}_1 = (\mathcal{A}_1, \beta)$ be an $S_1$-interpretation and $\mathcal{I}_2 = (\mathcal{A}_2, \beta)$ be an $S_2$-interpretation. both with the same domain $A_1 = A_2$. Put $S := S_1 \cap S_2$.*

- *Let $t$ be an $S$-term. If $\mathcal{I}_1$ and $\mathcal{I}_2$ agree on the $S$-symbols occuring in $t$ and on the variables occuring in $t$, then $\mathcal{I}_1(t) = \mathcal{I}_2(t)$.*
- *Let $\varphi$ be an $S$-formula. If $\mathcal{I}_1$ and $\mathcal{I}_2$ agree on the $S$-symbols and the variables occuring free in $\varphi$, then ($\mathcal{I}_1 \models \varphi$ iff. $\mathcal{I}_2 \models \varphi$).*

# Coincidence

# Coincidence

Coincidence lemma says that, for an $S$-formula $\varphi$ and an $S$-interpretation $\mathcal{I} = (\mathcal{A}, \beta)$, the validity of $\varphi$ under $\mathcal{I}$ depends only on the assignments for the *finitely many* variables occuring free in $\varphi$.
If these variables among $v_0, v_1, \ldots, v_{n-1}$, the $\beta$-values $a_i = \beta(v_i)$ for $i = 0, \ldots, n-1$ which are significant. Thus instead of $(\mathcal{A}, \beta) \models \varphi$, we shall often use the more suggestive notation

$$\mathcal{A} \models \varphi[a_0, \ldots, a_{n-1}] \tag{5}$$

If $\varphi$ is a sentence, we can choose $n = 0$ and write $\mathcal{A} \models \varphi$ without even mentioning an assignment. We say that $\mathcal{A}$ is a model of $\varphi$.

# Reduct and Expansion

## Definition

Let $S$ and $S'$ be a symbol sets such that $S \subseteq S'$. Let $\mathcal{A} = (A, \mathfrak{a})$ be an $S$-structure, and $\mathcal{A}' = (A', \mathfrak{a})$ be an $S'$-structure. we call $\mathcal{A}$ *a reduct* (or the $S$-reduct) of $\mathcal{A}'$ and write $\mathcal{A} = \mathcal{A}'|_S$ iff $A = A'$ and $\mathfrak{a}$ and $\mathfrak{a}'$ agrees on $S$. We say that $\mathcal{A}'$ is an *expansion* of $\mathcal{A}$.

## Example

The ordered field $\mathcal{R}^<$ of real numbers as an $S_{\mathrm{ar}}^<$-structure is an expansion of the field $\mathcal{R}$ of real numbers as $S_{\mathrm{ar}}$-structure.

$$\mathcal{R} = \mathcal{R}^<|_{S_{\mathrm{ar}}}$$

## Satisfiability on Reduct and Expansion

$\Phi$ is satisfiable with respect to $S$ iff $\Phi$ is satisfiable with respect to $S'$.

# Isomorphism

## Definition

Let $\mathcal{A}$ and $\mathcal{B}$ be $S$-Structures

- A map $\pi : A \to B$ is called an *isomorphism* of $\mathcal{A}$ onto $\mathcal{B}$ ($\pi : \mathcal{A} \simeq \mathcal{B}$) iff
  - $\pi$ is a bijection of $A$ onto $B$.
  - For $n$-ary $R \in S$ and $a_1, \ldots, a_n \in A$,

$$R^{\mathcal{A}}(a_1, \ldots, a_n) \quad \text{iff} \quad R^{\mathcal{B}}(\pi(a_1), \ldots, \pi(a_n))$$

  - For $n$-ary $f \in S$ and $a_1, \ldots, a_n \in A$,

$$\pi(f^{\mathcal{A}}(a_1, \ldots, a_n)) = f^{\mathcal{B}}(\pi(a_1), \ldots, \pi(a_n))$$

  - For $c \in S$,

$$pi(c^{\mathcal{A}}) = c^{\mathcal{B}}$$

- Structure $\mathcal{A}$ and $\mathcal{B}$ are said to be *isomorphic* ($\mathcal{A} \simeq \mathcal{B}$) iff. there is an isomorphism $\pi : \mathcal{A} \simeq \mathcal{B}$.

# Example of Isomorphism

## Example

The $S_{\mathrm{ar}}$-structure $(\mathbb{N}, +, 0)$ is isomorphic to the $S_{\mathrm{gr}}$-structure $(G, +^G, 0)$ consisting of the even natural numbers with ordinary addition $+^G$ and 0.

**Question**: Find the isomorphism $\pi : \mathbb{N} \to G$.

# Isomorphism

## Lemma

*For isomorphic $S$-structures $\mathcal{A}$ and $\mathcal{B}$ and every $S$-sentence $\varphi$,*

$$\mathcal{A} \models \varphi \quad \textit{iff.} \quad \mathcal{B} \models \varphi$$

## Corollary

*If $\pi : \mathcal{A} \simeq \mathcal{B}$, then for $\varphi \in L_n^S$ and $a_0, \ldots, a_{n-1} \in A$,*

$$\mathcal{A} \models \varphi[a_0, \ldots, a_{n-1}] \quad \textit{iff.} \quad \mathcal{B} \models \varphi[\pi(a_0), \ldots, \pi(a_{n-1})]$$

Note that, isomorphic structures cannot be distinguished in $L_0^S$. For example, there are structures not isomorphic to the $S_{\mathrm{ar}}$-structure $\mathcal{N}$ of natural numbers in which are the same first-order sentences hold.

# Substructure

## Definition

Let $\mathcal{A}$ and $\mathcal{B}$ be $S$-structures. Then $\mathcal{A}$ is called a *substructure* of $\mathcal{B}$ ($\mathcal{A} \subseteq \mathcal{B}$) iff.

- $A \subseteq B$
- 
  - for $n$-ary $R \in S$, $R^{\mathcal{A}} = R^{\mathcal{B}} \cap A^n$
  - for $n$-ary $f \in S$, $f^{\mathcal{A}}$ is the restriction of $f^{\mathcal{B}}$ to $A^n$
  - for $c \in S$, $c^{\mathcal{A}} = c^{\mathcal{B}}$

For example the $(\mathbb{Z}, +, 0)$ is a substructure of $(\mathbb{Q}, +, 0)$

# Substructure

## Lemma

*Let $\mathcal{A}$ and $\mathcal{B}$ be $S$-structures with $\mathcal{A} \subseteq \mathcal{B}$ and let $\beta : \{v_n | n \in \mathbb{N}\} \to A$ be an assignment in $\mathcal{A}$. Then the following holds for every $S$-term $t$*

$$(\mathcal{A}, \beta)(t) = (\mathcal{B}, \beta)(t)$$

*and for every quantifier-free $S$-formula $\varphi$:*

$$(\mathcal{A}, \beta) \models \varphi \quad \textit{iff.} \quad (\mathcal{B}, \beta) \models \varphi$$

# Universal Formula



## Definition

The formulas which are derivable by means of the following calculus are called *universal formulas*.

$$\frac{}{\varphi} \text{ if } \varphi \text{ is quantifier-free}$$

$$\frac{\varphi, \psi}{(\varphi \star \psi)} \text{ for } \star \in \{\wedge, \vee\}$$

$$\frac{\varphi}{\forall x \varphi}$$

# Substructure and Universal Formulas

## Lemma

*Let $\mathcal{A}$ and $\mathcal{B}$ be $S$-structures with $\mathcal{A} \subseteq \mathcal{B}$ and let $\varphi \in L_n^S$ be a universal. Then the following holds for all $a_0, \ldots, a_{n-1} \in A$:*

$$\text{If } \mathcal{B} \models \varphi[a_0, \ldots, a_{n-1}], \text{ then } \mathcal{A} \models \varphi[a_0, \ldots, a_{n-1}]$$

## Corollary

*If $\mathcal{A} \subseteq \mathcal{B}$, then the following holds for every universal sentence $\varphi$:*

$$\text{If } \mathcal{B} \models \varphi, \text{ then } \mathcal{A} \models \varphi$$

# Formalization of Group

## Definition

The axioms of group are the following formulas in $\Phi_{\mathrm{gr}}$:

$$\forall v_0 \forall v_1 \forall v_2 (v_0 \cdot (v_1 \cdot v_2) \equiv (v_0 \cdot v_1) \cdot v_2)$$
$$\forall v_0 (v_0 \cdot e \equiv v_0)$$
$$\forall v_0 \exists v_1 (v_1 \cdot v_0 \equiv e)$$

We can assign the set of mathematical objects to our structure, then we have interpretation of group $\mathcal{G}$.
**Question**: Formulate following sentences in first-order logic. "There is no element of order two in a group." (order two means that $a \circ a \equiv e$)
$\varphi := \neg \exists v_0 (\neg v_0 \equiv e \wedge v_0 \circ v_0 \equiv e)$

# Equivalence Relations

When the tree defining properties of an equivalence relation can be formalized with the aid of a single binary relation symbol $R$ as follows:

$$\forall x Rxx$$

$$\forall x \forall y (Rxy \to Ryx)$$

$$\forall x \forall y \forall z ((Rxy \land Ryz) \to Rxz)$$

**Question**: Formalize following example

### Example

If $x$ and $y$ are both equivalent to a third element, then they are equivalent to the same elements.

# Formalization of Field

The axioms of field are the following formulas in the symbol set $S_{\mathrm{ar}} = +, \cdot, 0, 1$:

$$\forall v_0 \forall v_1 \forall v_2 (v_0 + (v_1 + v_2) \equiv (v_0 + v_1) + v_2)$$
$$\forall v_0 (v_0 + 0 \equiv v_0)$$
$$\forall v_0 \exists v_1 (v_0 + v_1 \equiv 0)$$
$$\forall v_0 \forall v_1 (v_0 \cdot (v_1 \cdot v_2) \equiv (v_0 \cdot v_1) \cdot v_2)$$
$$\forall v_0 (v_0 \cdot 1 \equiv v_0)$$
$$\forall v_0 (\neg x \equiv 0 \rightarrow \exists y x \cdot y \equiv 1)$$
$$\forall v_0 \forall v_1 (v_0 + v_1 \equiv v_1 + v_0)$$
$$\forall v_0 \forall v_1 (v_0 \cdot v_1 \equiv v_1 \cdot v_0)$$
$$\forall v_0 \forall v_1 \forall v_2 ((v_0 + v_1) \cdot v_2 \equiv (v_0 \cdot v_2) + (v_1 \cdot v_2))$$
$$\neg 0 \equiv 1$$

# Limitation of First-order Logic: Torsion Group

A group $\mathcal{G}$ is called a *torsion group* if every element of $\mathcal{G}$ has finite order.
If for every $a \in G$ there is an $n \geq 1$ such that $a^n = e^G$.
**Question**: Can we add axioms of torsion group to our first-order logic?
**Answer**: No. We may "ad hoc" formulation of above statement as follow.

$$\forall x(x \equiv e \vee x \circ x \equiv e \vee x \circ x \circ x \equiv e \vee \ldots)$$

But our first-order logic cannot express the infinite disjunction.

# Limitation of First-order Logic: Peano's axioms

We discuss with the structure of natural number arithmetic system with addition as $\mathcal{N}_\sigma = (\mathbb{N}, \sigma, 0)$, where $\sigma$ is a unary successor function. Later, we may extend this structure to $\mathcal{N} = (\mathbb{N}, +, \cdot, 0, 1, <)$.

## Definition

$\mathcal{N}_\sigma$ satisfies the so-called Peano axiom system:

- 0 is not a value of the succesor function $\sigma$.
- $\sigma$ is injective.
- For every subset $X$ of $\mathbb{N}$, if $0 \in X$ and $\sigma(X) \subseteq X$, then $X = \mathbb{N}$.

**Question**: Can we formalize the Peano's axioms in first-order logic?
**Answer**: No. We may "ad hoc" formulation of above statement as next slide.

# Limitation of First-order Logic: Peano's axioms

$$\forall x \neg \sigma x \equiv 0$$
$$\forall x \forall y (\sigma x \equiv \sigma y \to x \equiv y)$$

How about third axiom?

$$\forall X \left( X0 \wedge \forall x (Xx \to X\sigma x) \to \forall y X y \right)$$

In third axiom, we need quantifier in set and quantifier on set which is not in first-order logic.

# Limitation of First-order Logic: Peano's axioms

### Theorem

*Every structure* $\mathcal{A} = (A, \sigma^A, 0^A)$ *which satisfies three Peano's axioms is isomorphic to* $\mathcal{N}_\sigma$.

In order to show that every element of the domain $A$ has a certain property $P$, one verifies that $0^A$ has the property $P$ and that if an element $a$ has the property $P$, then $\sigma^A(a)$ also has the property $P$. Suppose $\mathcal{A} = (A, \sigma^A, 0^A)$ is a structure which satisfies the Peano's axioms.

# Limitation of First-order Logic: Peano's axioms

The isomorphism $\pi : \mathcal{N}_\sigma \simeq \mathcal{A}$ is defined by induction on terms.

$$\pi(0) := 0^A$$
$$\pi(\sigma^{\mathbb{N}}(n)) := \sigma^A(\pi(n)) \text{ for all } n \in \mathbb{N}$$

that is

$$\pi(0) = 0^A \tag{6}$$
$$\pi(n+1) = \sigma^A(\pi(n)) \text{ for all } n \in \mathbb{N} \tag{7}$$

And we want to show that $\pi$ is a bijective map from $\mathbb{N}$ onto our domain $A$.

# Limitation of First-order Logic: Peano's axioms

**Surjectivity** By induction in $\mathscr{A}$ we prove that every element $A$ lies in the range of $\pi$.

**Injectivity** By induction on $n$ we want to prove "*For all $m \in \mathbb{N}$, if $m \neq n$, then* $\pi(m) \neq \pi(n)$."

# Substitution

Let's consider following example.

$$\varphi := \exists z z + z \equiv x$$

**Question**: If we replace the variable $x$ by $y$, then what is the result?
**Question**: If we replace the variable $x$ by $z$, then what is the result?
**Question**: If we replace the variable $z$ by $u$, then what is the result?
By above example, we need a rule for substitution. In process of replace, we need to replace only one variable at a time. In this section, we consider *pairwise distinct variables* $x_0, \ldots, x_r$ and arbitrary terms $t_0, \ldots, t_r$.
Note that, we can only define substitution when $x_i$ is occur free in formula.

# Substitution in Terms

In this section, we may wonder about how to define subtitute a term $t$ for a variable $x$ in a formula $\varphi$ at the places where $x$ occurs free.

## Definition

$$x\frac{t_0 \dots t_r}{x_0 \dots x_r} := \begin{cases} x & \text{if } x \neq x_0, \dots, x \neq x_r \\ t_i & \text{if } x = x_i \end{cases}$$

$$c\frac{t_0 \dots t_r}{x_0 \dots x_r} := c$$

$$[ft_1' \dots t_n']\frac{t_0 \dots t_r}{x_0 \dots x_r} := f\left(t_1'\frac{t_0 \dots t_r}{x_0 \dots x_r}, \dots, t_n'\frac{t_0 \dots t_r}{x_0 \dots x_r}\right)$$

# Substitution in Formulas

## Definition

$$[t_1' \equiv t_2'] \frac{t_0 \ldots t_r}{x_0 \ldots x_r} := t_1' \frac{t_0 \ldots t_r}{x_0 \ldots x_r} \equiv t_2' \frac{t_0 \ldots t_r}{x_0 \ldots x_r}$$

$$[Rt_1' \ldots t_n'] \frac{t_0 \ldots t_r}{x_0 \ldots x_r} := Rt_1' \frac{t_0 \ldots t_r}{x_0 \ldots x_r} \ldots t_n' \frac{t_0 \ldots t_r}{x_0 \ldots x_r}$$

$$[\neg\varphi] \frac{t_0 \ldots t_r}{x_0 \ldots x_r} := \neg \left[ \varphi \frac{t_0 \ldots t_r}{x_0 \ldots x_r} \right]$$

$$(\varphi \vee \psi) \frac{t_0 \ldots t_r}{x_0 \ldots x_r} := \left( \varphi \frac{t_0 \ldots t_r}{x_0 \ldots x_r} \vee \psi \frac{t_0 \ldots t_r}{x_0 \ldots x_r} \right)$$

How about quantifier?

# Substitution in Formulas

## Definition

Suppose $x_{i_1}, \dots, x_{i_s}$ $(i_1 < \cdots < i_s)$ are exactly the variables $x_i$ among the $x_0, \dots, x_r$ such that

$$x_i \in \mathrm{free}(\exists x \varphi) \quad \text{and} \quad x_i \neq t_i$$

In particular, $x \neq x_{i_1}, \dots, x \neq x_{i_s}$. Then set

$$[\exists x \varphi]\frac{t_0 \dots t_r}{x_0 \dots x_r} := \exists u \left[\varphi \frac{t_{i_1} \dots t_{i_s} u}{x_{i_1} \dots x_{i_s} x}\right]$$

where $u$ is the variable $x$ if $x$ does not occur in $t_{i_1} \dots t_{i_s}$, otherwise $u$ is the first variable in the list $v_0, v_1, v_2, \dots$ which does not occur in $\varphi, t_{i_1}, \dots, t_{i_s}$.

# Substitution Lemma

## Lemma

*For every term t*

$$\mathcal{I}(t\frac{t_0 \ldots t_r}{x_0 \ldots x_r}) = \mathcal{I}\frac{\mathcal{I}(t_0), \ldots, \mathcal{I}(t_r)}{x_0, \ldots, x_r}(t)$$

*And for every formula* $\varphi$

$$\mathcal{I} \models \varphi\frac{t_0 \ldots t_r}{x_0 \ldots x_r} \quad \textit{iff} \quad \mathcal{I}\frac{\mathcal{I}(t_0), \ldots, \mathcal{I}(t_r)}{x_0, \ldots, x_r} \models \varphi$$

# Rank of Formula

The number of connectives and quantifiers occurring in a formula $\varphi$ the *rank* of $\varphi$, written $\mathrm{rk}(\varphi)$.
**Question**: How can we define it?
**Question**: After substitution, the rank of formula is changed?