Mathematical Logic and Computability

Lecture 4: Sequent Calculus

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Introduction to Sequent Calculus

In this lecture, we will learn about *Sequent Calculus* to develop a proof system for first-order logic.

To show that a proposition follows from the axioms, we need to supply a proof like follows:

$$\Phi \to \varphi_1 \to \ldots \to \varphi_n \to \varphi$$

where the set of formulas Φ is the set of axioms and φ is the proposition to be proved.

How can we show that φ follows from Φ? Also, how can we concrete the proof?

For convience, we fix a symbol set as S.

Sequent Rules

Our observation in proof, we can describe a proof by listing the corresponding assumptions and the respective claim. If we call a nonempty list (sequence) of formulas a *sequent*, then we can use sequents to describe ”stage of a proof”.

The stages with assumptions are notated as $\varphi_1, \ldots, \varphi_n$ and our claim φ is randered by the sequent $\varphi_1 \ldots \varphi_n \varphi$.
The sequence $\varphi_1 \ldots \varphi_n$ is called the *antecedent* and φ is called the *succedent* of the sequent.

Sequent Rules

Remind the Sequent Calculi notation in the first lecture.
Let's consider the proof by contradiction as follows:

$$\begin{array}{ccc} \varphi_1 \dots \varphi_n & \neg\varphi & \psi \\ \varphi_1 \dots \varphi_n & \neg\varphi & \neg\psi \\ \hline \varphi_1 \dots \varphi_n & & \varphi \end{array}$$

We can say that "Our assumptions $\varphi_1 \dots \varphi_n$ and the negation of claim $\neg\varphi$ lead to a contradiction $\psi \wedge \neg\psi$. "
For convience, let's use the letters $\Gamma, \Delta, \dots$ to denote finite sets of formulas. (of course, possibly empty) If we define $\Gamma = \varphi_1 \dots \varphi_n$, we can rewrite as follows:

$$\begin{array}{ccc} \Gamma & \neg\varphi & \psi \\ \Gamma & \neg\varphi & \neg\psi \\ \hline \Gamma & & \varphi \end{array}$$

Derivable

If in the calculus $\mathcal{C}$, there is a derivation of the sequent $\Gamma\varphi$, then we write $\vdash \Gamma\varphi$ and say that $\Gamma\varphi$ is *derivable*.

Definition

A formula φ is *formally provable* or *derivable* from a set Φ of formulas ($\Phi \vdash \varphi$), if and only if there are finitely many formulas $\varphi_1, \ldots, \varphi_n$ in Φ such that $\vdash \varphi_1 \ldots \varphi_n\varphi$

Antecedent and Assumption Rules

Antecedent Rule (Ant)

if every member of Γ is also a member of Γ' ($\Gamma \subseteq \Gamma'$)

$$\frac{\Gamma \quad \varphi}{\Gamma' \quad \varphi} \tag{1}$$

Assumption Rule (Assm)

if φ is a member of Γ

$$\frac{}{\Gamma \quad \varphi} \tag{2}$$

Question, show that correctness of the antecedent rule and the assumption rule.

Proof by Case and Contradiction Rules

Proof by Case Rule (PC)

$$\begin{array}{lll} \Gamma & \psi & \varphi \\ \Gamma & \neg\psi & \varphi \\ \hline \Gamma & & \varphi \end{array} \tag{3}$$

Proof by Contradiction (Ctr)

$$\begin{array}{lll} \Gamma & \neg\varphi & \psi \\ \Gamma & \neg\varphi & \neg\psi \\ \hline \Gamma & & \varphi \end{array} \tag{4}$$

Rule for Disjunction

∨-Rule for the Antecedent (∨A)

$$\begin{array}{lcc} \Gamma & \varphi & \xi \\ \Gamma & \psi & \xi \\ \hline \Gamma & (\varphi \vee \psi) & \xi \end{array} \tag{5}$$

∨-Rule for the Succedent (∨S)

$$\begin{array}{lc} \Gamma & \varphi \\ \hline \Gamma & (\varphi \vee \psi) \end{array} \qquad \begin{array}{lc} \Gamma & \varphi \\ \hline \Gamma & (\psi \vee \varphi) \end{array} \tag{6}$$

Chain Rule

Second Contradiction Rule (Ctr')

$$\frac{\begin{array}{ll} \Gamma & \psi \\ \Gamma & \neg\psi \end{array}}{\begin{array}{ll} \Gamma & \varphi \end{array}} \tag{7}$$

Chain Rule (Ch)

$$\frac{\begin{array}{lll} \Gamma & & \varphi \\ \Gamma & \varphi & \psi \end{array}}{\begin{array}{lll} \Gamma & & \psi \end{array}} \tag{8}$$

Equation 7 looks very weired. Question: How can we prove these?

Contraposition Rules

Contraposition Rules (Cp)

$$\frac{\Gamma \quad \varphi \quad \psi}{\Gamma \quad \neg\psi \quad \neg\varphi} \qquad \frac{\Gamma \quad \neg\varphi \quad \neg\psi}{\Gamma \quad \psi \quad \varphi} \tag{9}$$

$$\frac{\Gamma \quad \neg\varphi \quad \psi}{\Gamma \quad \neg\psi \quad \varphi} \qquad \frac{\Gamma \quad \varphi \quad \neg\psi}{\Gamma \quad \psi \quad \neg\varphi} \tag{10}$$

Modus Ponens

Proposition

$$\begin{array}{ll} \Gamma & (\varphi \vee \psi) \\ \Gamma & \neg\varphi \\ \hline \Gamma & \psi \end{array} \tag{11}$$

Modus Ponens (MP)

$$\begin{array}{ll} \Gamma & (\varphi \rightarrow \psi) \\ \Gamma & \varphi \\ \hline \Gamma & \psi \end{array} \tag{12}$$

Rule for Universal Quantifier

Rule for ∃-Introduction in the Succedent (∃ S)

$$\frac{\Gamma \quad \varphi\frac{t}{x}}{\Gamma \quad \exists x \varphi} \tag{13}$$

Rule for ∃-Introduction in the Antecedent (∃ A)

$$\frac{\Gamma \quad \varphi\frac{y}{x} \quad \psi}{\Gamma \quad \exists x \varphi \quad \psi} \tag{14}$$

Rule for Equality

Figure: Equivalence of Equality

Reflexivity Rule for Equality ($\equiv$)

$$\frac{}{t \equiv t} \tag{15}$$

Rule for Equality

Figure: Equivalence of Equality

Substitution Rule for Equality (Sub)

$$\frac{\Gamma \qquad\qquad \varphi\frac{t}{x}}{\Gamma \quad t \equiv t' \quad \varphi\frac{t'}{x}} \tag{16}$$

Further Derivable Rules

$$\frac{\Gamma \quad \varphi}{\Gamma \quad \exists x\varphi} \qquad \frac{\Gamma \quad \varphi \quad \psi}{\Gamma \quad \exists x\varphi \quad \psi} \quad \text{if } x \text{ is not free in } \Gamma\psi \tag{17}$$

$$\frac{\Gamma \qquad\qquad \varphi}{\Gamma \quad x \equiv t \quad \varphi\frac{t}{x}} \tag{18}$$

$$\frac{\Gamma \quad t_1 \equiv t_2}{\Gamma \quad t_2 \equiv t_1} \qquad \frac{\begin{array}{ll}\Gamma & t_1 \equiv t_2 \\ \Gamma & t_2 \equiv t_3\end{array}}{\Gamma \quad t_1 \equiv t_3} \tag{19}$$

$$\frac{\begin{array}{ll}\Gamma & Rt_1 \ldots t_n \\ \Gamma & t_1 \equiv t_1' \\ \vdots & \vdots \\ \Gamma & t_n \equiv t_n'\end{array}}{\Gamma \quad Rt_1' \ldots t_n'} \qquad \frac{\begin{array}{ll}\Gamma & t_1 \equiv t_1' \\ \vdots & \vdots \\ \Gamma & t_n \equiv t_n'\end{array}}{\Gamma \quad ft_1 \ldots t_n \quad ft_1' \ldots t_n'} \tag{20}$$

Correctness

Lemma

For all Φ and φ, if and only if there is a finite subset Φ_0 of Φ such that $\Phi_0 \vdash \varphi$.

Soundness Theorem

For all Φ and φ, if $\Phi \vdash \varphi$, then $\Phi \models \varphi$.

Inverse of the theorem (completeness) is shown in the next lecture.

Exercise

Prove formal proof of the existence of left inverse from the group axioms as following

$$\begin{aligned}
\varphi_0 &: \forall x \forall y \forall z (x \circ y) \circ z \equiv x \circ (y \circ z) \\
\varphi_1 &: \forall x x \circ e \equiv x \\
\varphi_2 &: \forall x \exists y x \circ y \equiv e
\end{aligned}$$

Consistency

Definition

$\mathrm{Con}\Phi$: A set Φ is *consistent* if and only if there is no formula φ such that $\Phi \vdash \varphi$ and $\Phi \vdash \neg\varphi$.

$\mathrm{Inc}\Phi$: A set Φ is *inconsistent* if and only if it is not consistent.

Consistency Lemmas

Lemma

For a set of formulas Φ the following are equivalent:

- *Φ is inconsistent.*
- *For all φ, $\Phi \vdash \varphi$.*

Lemma

For a set of formulas Φ the following are equivalent:

- *Φ is consistent.*
- *There is a formula φ which is not derivable from Φ.*

Consistency Lemmas

In next lecture, we will prove the completeness theorem, which states that for all Φ and φ, $\Phi \vdash \varphi$ if and only if $\Phi \models \varphi$.

Lemma

For all Φ, $Con\Phi$ if and only if $Con\Phi_0$ for all finite subsets Φ_0 of Φ.

Lemma

Every satisfiable set of formula is consistent

Lemma

For all Φ and φ the following holds:

- $\Phi \vdash \varphi$ *if and only if* $Inc\Phi \cup \{\neg\varphi\}$.
- $\Phi \vdash \neg\varphi$ *if and only if* $Inc\Phi \cup \{\varphi\}$.
- *If* $Con\Phi$, *then* $Con\Phi \cup \{\varphi\}$ *or* $Con\Phi \cup \{\neg\varphi\}$.

Consistency Lemmas

Lemma

For $n \in \mathbb{N}$ let S_n be symbol sets such that

$$S_0 \subseteq S_1 \subseteq S_2 \subseteq \dots$$

and let Φ_n be set of S_n-formulas such that $\mathcal{Con}_{S_n}\Phi_n$ and

$$\Phi_0 \subseteq \Phi_1 \subseteq \Phi_2 \subseteq \dots$$

let $S = \cup_{n\in\mathbb{N}} S_n$ and $\Phi = \cup_{n\in\mathbb{N}} \Phi_n$. Then $\mathcal{Con}_S\Phi$.