Mathematical Logic and Computability

Lecture 5: Completeness Theorem

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Henkin's Theorem

Satisfiability of Consistent Sets of Countable Formulas

Completeness Theorem

What is Completeness Theorem?

In previous lecture, we learn about the sequent calculus, which is a formal system for first-order logic. Also, we proved the soundness theorem for sequent calculus.

Soundness Theorem

If $\Gamma \vdash \phi$, then $\Gamma \models \phi$.

In this lecture, we will prove the completeness theorem for first-order logic.

Completeness Theorem

If $\Gamma \models \phi$, then $\Gamma \vdash \phi$.

We may wonder about, there is a statement such that consequence of Γ but not provable from Γ. We need to prove two system are equivalent, i.e., $\Gamma \vdash \phi$ iff $\Gamma \models \phi$.

From Semantics to Syntactics

We want to prove from semantical statement to syntactical statement.
Let's Φ be a consistent set of formulas. There is an interpretation $\mathcal{I} = (\mathcal{A}, \beta)$ such that satisfy Φ. We want to obtain a model using syntatical objects.
First, we take a domain A that set T^S of all S-terms, to define β as follows:

$$\beta(v_i) := v_i \quad \text{for} i \in \mathbb{N}$$

And define unary function and relation simbol f and R as follows:

$$f^{\mathcal{A}}(t) := ft \quad \text{and} \qquad R^{\mathcal{A}} := \{t \in A | \Phi \vdash Rt\}$$

We can have statement $\mathcal{I}(fx) = f^{\mathcal{A}}(\beta(x))$.
Question: Is there any problem with this definition?
Question: How about y is a variable different from x? $fx \neq fy$, then $\mathcal{I}(fx) \neq \mathcal{I}(fy)$.

From Semantics to Syntactics

In previous quetsion, if we choose Φ such that $\Phi \vdash fx \equiv fy$ $\mathcal{I}$ is not a model of Φ.
We need to *correct* this interpretation gap.
We define an interpretation $\mathcal{I}^{\Phi} = (\mathcal{T}^{\Phi}, \beta^{\Phi})$. We introduce a binary relation $\sim$ on the set T^S of S-terms as follows

$$t_1 \sim t_2 \quad \text{iff} \quad \Phi \vdash t_1 \equiv t_2 \tag{1}$$

Lemma 1

1. The relation $\sim$ (equation 1) is an equivalence relation on T^S.
2. $\sim$ is compatible with the simbols in S in the follows:
If $t_1 \sim t'_1, \ldots, t_n \sim t'_n$, then for n-ary function $f \in S$ and n-ary relation $R \in S$

$$ft_1 \ldots t_n \sim ft'_1 \ldots t'_n \quad \Phi \vdash Rt_1 \ldots t_n \textit{ iff } \Phi \vdash Rt'_1 \ldots t'_n$$

Term Structure

Let $\bar{t}$ be the equivalence class of t:

$$\bar{t} := \{t' \in T^S | t \sim t'\}$$

and let T^Φ be the set of equivalence class:

$$T^\Phi := \{\bar{t} | t \in T^S\}$$

We define the S-structure $\mathcal{T}^\Phi$ ovwer T^Φ, the so-called *term structure* corresponding to Φ.

Term Interpretation

We need to define the interpretation of term structure $\mathcal{T}^{\Phi}$.
For n-ary relation $R \in S$ and n-ary function $f \in S$, we define

$$R^{\mathcal{T}^{\Phi}}(\overline{t_1}, \ldots, \overline{t_n}) \quad \text{iff} \quad \Phi \vdash Rt_1 \ldots t_n$$
$$f^{\mathcal{T}^{\Phi}}(\overline{t_1}, \ldots, \overline{t_n}) := \overline{ft_1 \ldots t_n}$$

For constant and variable, we define

$$c^{\mathcal{T}^{\Phi}} := \overline{c}$$
$$\beta^{\Phi}(x) := \overline{x}$$

We call the *term interpretation* $\mathcal{I}^{\Phi} := (\mathcal{T}^{\Phi}, \beta^{\Phi})$

Term Interpretation

Lemma 2

For all t, $\mathcal{I}^{\Phi}(t) = \bar{t}$.

For every atomic formula φ,

$$\mathcal{I}^{\Phi} \models \varphi \quad \textit{iff} \quad \Phi \vdash \varphi$$

For every formula φ *and pairwise distinct variables* $x_1, \ldots, x_n$,

$$\mathcal{I}^{\Phi} \models \exists x_1 \ldots \exists x_n \varphi \quad \textit{iff} \quad \textit{there are } t_1, \ldots, t_n \in T^S \textit{ with } \mathcal{I}^{\Phi} \models \varphi \frac{t_1 \ldots t_n}{x_1 \ldots x_n}$$

$$\mathcal{I}^{\Phi} \models \forall x_1 \ldots \forall x_n \varphi \quad \textit{iff} \quad \textit{for all terms } t_1, \ldots, t_n \in T^S, \mathcal{I}^{\Phi} \models \varphi \frac{t_1 \ldots t_n}{x_1 \ldots x_n}$$

Negation Complete and Contains Witnesses

Negation Complete

Φ is *negation complete* iff for every formula φ,

$$\Phi \vdash \varphi \text{ or } \Phi \vdash \neg\varphi.$$

Contains witnesses

Φ *contains witnesses* iff for every formula of the form $\exists x \varphi$ there exists a term t such that $\Phi \vdash (\exists x \varphi \to \varphi \frac{t}{x})$

Lemma on Consistent and Negation Complete Sets

Lemma 3

Suppose that Φ is consistent and negation complete and that it contains withness. Then the following holds for all φ and ψ

$$\Phi \vdash \neg\varphi \quad \textit{iff} \quad \textit{not} \Phi \vdash \varphi \tag{2}$$

$$\Phi \vdash (\varphi \vee \psi) \quad \textit{iff} \quad \Phi \vdash \varphi \textit{ or } \Phi \vdash \psi \tag{3}$$

$$\Phi \vdash \exists x\varphi \quad \textit{iff} \quad \textit{there is a term } t \textit{ with } \Phi \vdash \varphi\frac{t}{x} \tag{4}$$

Henkin's Theorem

Theorem 4

Henkin's Theorem Let Φ be a consistent set of formulas which is negation complete and contains witnesses. Then for all φ,

$$\mathcal{I}^{\Phi} \models \varphi \quad \textit{iff} \quad \Phi \vdash \varphi$$

Lemma 5

If Φ is a consistent set which is negation complete and contains witnesses, then $\mathcal{I}^{\Phi}$ is a model of Φ.

Proof Sketch

We prove theorem 4 by induction on the rank of φ. When $\mathrm{rk}(\varphi) = 0$, we apply lemma 2.

Satisfiability of Consistent Set of Formulas

By Henkin's theorem, we know that every consistent set of formulas which is negation complete and contains witnessesm, is satisfiable.
In this section, we expand this result to arbitrary consistent sets of formulas.
First, we show the case when *the symbol set S is countable*. When this assumption, we have only finitely many variables occur free in the consistent set Φ of formulas. $\mathrm{free}(\Phi) := \cup_{\varphi \in \Phi} \mathrm{free}(\varphi)$ is finite.

Two Lemma for Proof Sketch

Lemma 6

Let $\Phi \subseteq L^S$ be consistent and let $free(\Phi)$ be finite. Then there is a consistent set Ψ such that $\Phi \subseteq \Psi \subseteq L^S$ is contains witnesses.

Lemma 7

Let $\Psi \subseteq L^S$ be consistent. Then there is a consistent and negation complete set Θ such that $\Psi \subseteq \Theta \subseteq L^S$.

By the two lemma, we can prove the satisfiability of consistent set of formulas.

Lemma 6 : Construct Contains Witnesses

By our assumption the symbol set S is countable, the set of all formulas L^S is countable.
Let's make a list of all formulas with an existential quantifier $\exists x_0 \varphi_0, \exists x_1 \varphi_1, \ldots$.
Inductively we define formulas $\psi_0, \psi_1, \ldots$ which add to Φ. For each n, the formula ψ_n is a "*witness formula*" for $\exists x_n \varphi_n$.

Lemma 6 : Construct Contains Witnesses

Assume that we already defined ψ_m for $m < n$. We want to prove than we define witness formula ψ_n.
Since $\mathrm{free}(\Phi)$ is finite, only finitely many variables occur free in $\Phi \cup \{\psi_m | m < n\} \cup \{\exists x_n \varphi_n\}$.
We can choose an index n be the variable with smallest index distinct from these. We set witness formula as follows:

$$\psi_n := (\exists x_n \varphi_n \to \varphi_n \frac{y_n}{x_n})$$

Now let construct the set $\Psi := \Phi \cup \{\psi_n | n \in \mathbb{N}\}$. It contains witnesses.
Question: We prove done? **Nooo**. We need to prove Ψ is consistent.

Lemma 6 : Construct Contains Witnesses

Let define $\Phi_n := \Phi \cup \{\psi_m | m < n\}$. Then, $\Phi_0 \subseteq \Phi_1 \subseteq \ldots$.
and $\Psi = \cup_{n \in \mathbb{N}} \Phi_n$. By the last lemma in previous lecture, if we show Φ_n is consistent, the proof is done.
In base case $\Phi_0 = \Phi$ is consistent by assumption of lemma.
For induction, we assume that Φ_n is consistent. We want to prove Φ_{n+1} is consistent.
Let's proof by contradiction.

Lemma 6 : Construct Contains Witnesses

We assume that $\Phi_{n+1} = \Phi_n \cup \{\psi_n\}$ is inconsistent. Then, for any formula φ there exists Γ over Φ_n such that $\vdash \Gamma\psi_n\varphi$.

$$\vdash \Gamma(\neg\exists x_n\varphi_n \lor \varphi_n\frac{y_n}{x_n})\varphi$$

There is derivation of these. We extend this derivation as next slide.

Lemma 6 : Construct Contains Witnesses

$$
\begin{array}{rlllr}
m. & \Gamma & (\neg\exists x_n\varphi_n \vee \varphi_n\frac{y_n}{x_n})\varphi & & \\
(m+1). & \Gamma & \neg\exists x_n\varphi_n & \neg\exists x_n\varphi_n & \text{(Assm)} \\
(m+2). & \Gamma & \neg\exists x_n\varphi_n & (\neg\exists x_n\varphi_n \vee \varphi_n\frac{y_n}{x_n}) & \text{(}\vee\text{S) applied to (m+1).} \\
(m+3). & \Gamma & \neg\exists x_n\varphi_n & \varphi & \text{(Ch) applied to (m+2) and m} \\
 & \vdots & & & \\
l. & \Gamma & \varphi_n\frac{y_n}{x_n} & \varphi & \text{Analogously} \\
(l+1). & \Gamma & \exists x_n\varphi_n & \varphi & \text{(}\exists\text{A) applied to (l)} \\
(l+2). & \Gamma & & \varphi & \text{(PC) applied to (l+1) and (m+3)}
\end{array}
$$

Lemma 6 : Construct Contains Witnesses

If we set $\varphi = \exists v_0 v_0 \equiv v_0$ and for $\varphi = \neg\exists v_0 v_0 \equiv v_0$, this gives that $\Phi_n \to \exists v_0 v_0 \equiv v_0$ and $\Phi_n \to \neg\exists v_0 v_0 \equiv v_0$ are both provable from Φ_n. This contradicts the assumption that Φ_n is consistent.

Lemma 7 : Construct Negation Complete

Suppose Ψ is consistent and let's make enumeration of all formulas $\varphi_0, \varphi_1, \ldots \in L^S$.
We define set of formulas Θ_n inductively as follows:

$$\Theta_0 := \Psi$$
$$\Theta_{n+1} := \begin{cases} \Theta_n \cup \{\varphi_n\} & \text{if Con} \Theta_n \cup \{\varphi_n\} \\ \Theta_n & \text{otherwise,} \end{cases}$$

and $\Theta := \cup_{n \in \mathbb{N}} \Theta_n$.
By the last lemma of previous lecture, $\Psi \subseteq \Theta$, all Θ_n is consistent, Θ is consistent.

Lemma 7 : Construct Negation Complete

For if $\varphi \in L^S$, say $\varphi = \varphi_n$, and not $\Theta \vdash \neg\varphi$, then $\mathrm{Con}\Theta \cup \{\varphi\}$ and therefore $\mathrm{Con}\Theta_n \cup \{\varphi\}$. So $\Theta_{n+1} = \Theta_n \cup \{\varphi\}$, hence $\varphi \in \Theta$ and therefore $\Theta \vdash \varphi$.
Now we drop the assumetion $\mathrm{free}(\Phi)$ is finite.

Theorem 8

If S is at most countable and $\Phi \subseteq L^S$ is consistent, then Φ is satisfiable.

We need to reduce this theorem 8 to lemma 6 and lemma 7.
Main idea is the replacing the free variables by new constants.

Theorem 8 : Proof

Let $c_0, c_1, \ldots$ be new distinct constants which do not belong to S and the set $S' = S \cup \{c_0, c_1, \ldots\}$.
For $\varphi \in L^S$ denote by $n(\varphi)$ the smalles n with $\mathrm{free}(\varphi) \in \{v_0, \ldots, v_{n-1}\}$. Let

$$\varphi' := \varphi \frac{c_0 \ldots c_{n(\varphi)}}{v_0 \ldots v_{n(\varphi)}} \quad \text{and} \quad \Phi' := \{\varphi' | \varphi \in \Phi\}$$

First, when $\mathrm{free}(\Phi') = \emptyset$, Φ' is a set of S'-sentences.
Now we need to show that $\mathrm{Con}_{S'}\Phi'$.
We already know when number of free variable is finite. Let's the sentence Φ' is satisfiable by the interpetation $\mathcal{I}' = (\mathcal{A}', \beta')$.
By *Coincidence Lemma*, we can choose β' such that $\beta'(v_n) = c_n^{\mathcal{A}'}$ for every $n \in \mathbb{N}$. Then, we apply *Substitution Lemma* for $\varphi \in \Phi$ we have $\mathcal{I}' \models \varphi$, since $\mathcal{I}' \models \varphi \frac{c_0 \ldots c_{n(\varphi)-1}}{v_0 \ldots v_{n(\varphi)-1}}$.
We have the interpretation $\mathcal{I}'$ for Φ, then Φ is satisfiable.

Theorem 8 : Proof

We suffice to show that every finite subset $\Phi'_0 \subset \Phi'$ is satisfiable, and thus, the last lemma of previous lecture, the Φ is consistent. Let $\Phi'_0 = \varphi'_1, \ldots, \varphi'_n$, where $\varphi_1, \ldots, \varphi_n \in \Phi$. S Since $\{\varphi_1, \ldots, \varphi_n\}$ is a subset of Φ, it is consistent, and since only finitely many variables occur free in Φ'_0, we can choose an interpretation $\mathcal{I} = (\mathcal{A}, \beta)$ such that $\mathcal{I} \models \Phi'_0$.
and expand the domain $\mathcal{A}$ to an S'-structure $\mathcal{A}'$ with $c_i^{\mathcal{A}'} = \mathcal{I}(v_i)$ for $i \in \mathbb{N}$. For this new S'-interpretation $\mathcal{I}' = (\mathcal{A}', \beta)$ the Substitution Lemma show us for $\varphi \in L^S$

$$\mathcal{I} \models \varphi \quad \text{iff} \quad \mathcal{I}' \models \varphi \frac{c_0 \ldots c_{n(\varphi)-1}}{v_0 \ldots v_{n(\varphi)-1}}$$

Completeness Theorem

In uncountable set of symbols, beyond our scope, please refer to textbook.

Completeness Theorem

For $\Phi \subseteq L^S$ and $\varphi \in L^S$:

$$\text{If } \Phi \models \varphi, \text{ then } \Phi \vdash_S \varphi$$

Theorem 9

The set of formula Φ is consistent iff Φ is satisfiable.

The theorem 25 is the completeness theorem for first-order logic. It is proved by Gödel in 1928, then it called *Gödel's Completeness Theorem*.