Mathematical Logic and Computability **Fall 2023**

Semantics of First-order Logic

September 15, 2023 Yunpyo An

Source of this exercise sheet is *Mathematical Logic* by *Heinz-Dieter Ebbinghaus et al.*

1. For S-strucutres $\mathcal{A} = (A, \mathfrak{a})$ and $\mathcal{B} = (B, \mathfrak{b})$ let $\mathcal{A} \times \mathcal{B}$, the *direct product* of $\mathcal{A}$ and $\mathcal{B}$, be the S-structure with domain
$$A \times B := \{(a, b) | a \in A, b \in B\}$$
which is determined by the following conditions
for n-ary R in S and $(a_1, b_1), \ldots, (a_n, b_n) \in A \times B$
$$R^{\mathcal{A}\times\mathcal{B}}(a_1, b_1) \ldots (a_n, b_n) \quad \text{iff} \quad R^{\mathcal{A}}(a_1, \ldots, a_n) \text{ and } R^{\mathcal{B}}(b_1, \ldots, b_n)$$
for n-ary f in S and $(a_1, b_1), \ldots, (a_n, b_n) \in A \times B$
$$f^{\mathcal{A}\times\mathcal{B}}((a_1, b_1) \ldots (a_n, b_n)) = (f^{\mathcal{A}}(a_1, \ldots, a_n), f^{\mathcal{B}}(b_1, \ldots, b_n))$$
for $c \in C$ and $(a, b) \in S$
$$c^{\mathcal{A}\times\mathcal{B}} = (c^{\mathcal{A}}, c^{\mathcal{B}})$$
Question: Show following statements
   (a) If the S_{ar}-structures $\mathcal{A}$ and $\mathcal{B}$ are groups, then $\mathcal{A} \times \mathcal{B}$ is also a group.
   (b) If the S_{ar}-structures $\mathcal{A}$ and $\mathcal{B}$ are fields, then $\mathcal{A} \times \mathcal{B}$ is not a field.

2. Let I be a nonempty set. For every $i \in I$, let $\mathcal{A}_i$ be an S-structure. We write $\prod_{i\in I} \mathcal{A}_i$ for the *direct product* of the structures $\mathcal{A}_i$, that is, the S-structure $\mathcal{A}$ with domain
$$\prod_{i\in I} A_i := \{g | g : I \to \cup_{i\in I} A_i, \text{and for all } i \in I : g(i) \in A_i\}$$
which is determined by the following conditions. (where for $g \in \prod_{i\in I} A_i$ we also write $\langle g(i) | i \in I\rangle$) For n-ary $R \in S$ and $g_1, \ldots, g_n \in \prod_{i\in I} A_i$,
$$R^{\mathcal{A}} g_1 \ldots g_n \quad \text{iff} \quad \forall i \in I : R^{\mathcal{A}_i} g_1(i) \ldots g_n(i)$$
for n-ary $f \in S$ and $g_1, \ldots, g_n \in \prod_{i\in I} A_i$,
$$f^{\mathcal{A}}(g_1 \ldots g_n) = \langle f^{\mathcal{A}_i} g_1(i) \ldots g_n(i) | i \in I\rangle$$
and for $c^{\mathcal{A}} := \langle c^{\mathcal{A}_i} | i \in I\rangle$.
Question: If t is an S-term with var $\subseteq \{v_0, \ldots, v_{n-1}\}$ and if $g_0, \ldots, g_{n-1} \in \prod_{i\in I} A_i$, then the following holds:
$$t^{\mathcal{A}}[g_0, \ldots, g_{n-1}] = \langle t^{\mathcal{A}_i}[g_0(i), \ldots, g_{n-1}(i)] | i \in I\rangle$$

3. Formulas which are derivable in the following calculus are called *Horn formulas*

$$\frac{}{(\neg\varphi_1 \vee \ldots \vee \neg\varphi_n \vee \varphi)} \text{ if } n \in \mathbb{N} \text{ and } \varphi_1, \ldots, \varphi_n, \varphi \text{ are atomic formulas}$$

$$\frac{}{\neg\varphi_0 \vee \ldots \vee \neg\varphi_n} \text{ if } n \in \mathbb{N} \text{ and } \varphi_1, \ldots, \varphi_n \text{ are atomic formulas}$$

$$\frac{\varphi, \psi}{(\varphi \wedge \psi)} \qquad \frac{\varphi}{\forall x \varphi} \qquad \frac{\varphi}{\exists x \varphi}$$

Horn formulas without free variables are called *Horn sentences.*
Question: If φ is a Horn sentence and if $\mathcal{A}_i$ is a model of φ for $i \in I$, then $\prod_{i \in I} \mathcal{A}_i \models \varphi$.
Note: If formula φ is derived from only terms it called *atomic formula.*

1 Definitions and Notations

1.1 Structure and Interpretation

Definition 1.1. An S-structure is a pair $\mathcal{A} = (A, \mathfrak{a})$ with the following properties:

- A is a non-empty set, called the *domain* or *universe* of $\mathcal{I}$
- $\mathfrak{a}$ is a function that assigns from symbols to following:
  - for every n-ary relation symbol R in S, $\mathfrak{a}(R)$ is an n-ary relation on A
  - for every n-ary function symbol f in S, $\mathfrak{a}(f)$ is an n-ary function on A
  - for every constant c in S, $\mathfrak{a}(c)$ is an element of A

Definition 1.2. An *assignment* in S-structure $\mathcal{A}$ is a function $\beta : \{v_n | n \in \mathbb{N}\} \to A$ from the set of variables into the domain A.

Definition 1.3. An S-interpretation $\mathcal{I}$ is a pair $(\mathcal{A}, \beta)$, where $\mathcal{A}$ is an S-structure and β is an assignment in $\mathcal{A}$.

1.2 Satisfication Relation

Definition 1.4. We define $\mathcal{I}(\varphi)$ by induction on terms

- For a variable x let $\mathcal{I}(x) = \beta(x)$
- For a constant $c \in S$ let $\mathcal{I}(c) = c^{\mathcal{A}}$
- For n-ary function symbol $f \in S$ and terms $t_1, \ldots, t_n$ let $\mathcal{I}(f(t_1, \ldots, t_n)) = f^{\mathcal{A}}(\mathcal{I}(t_1), \cdots, \mathcal{I}(t_n))$

Definition 1.5. For all interpretations $\mathcal{I} = (\mathcal{A}, \beta)$ we define following interpretations

- $\mathcal{I} \models (t_1 \equiv t_2)$ iff. $\mathcal{I}(t_1) = \mathcal{I}(t_2)$
- $\mathcal{I} \models (Rt_1 \ldots t_n)$ iff. $R^{\mathcal{A}}(\mathcal{I}(t_1), \ldots, \mathcal{I}(t_n))$
- $\mathcal{I} \models (\neg\varphi)$ iff. not $\mathcal{I} \models \varphi$
- $\mathcal{I} \models (\varphi \wedge \psi)$ iff. $\mathcal{I} \models \varphi$ and $\mathcal{I} \models \psi$
- $\mathcal{I} \models (\varphi \vee \psi)$ iff. $\mathcal{I} \models \varphi$ or $\mathcal{I} \models \psi$
- $\mathcal{I} \models (\varphi \to \psi)$ iff. $\mathcal{I} \models \varphi$ implies $\mathcal{I} \models \psi$
- $\mathcal{I} \models (\varphi \leftrightarrow \psi)$ iff. $\mathcal{I} \models \varphi$ iff. $\mathcal{I} \models \psi$
- $\mathcal{I} \models (\forall x\varphi)$ iff. for all $a \in A$, $\mathcal{I}\frac{a}{x} \models \varphi$
- $\mathcal{I} \models (\exists x\varphi)$ iff. there exists $a \in A$, $\mathcal{I}\frac{a}{x} \models \varphi$

1.3 Consequence Relation

Definition 1.6. Let Φ be a set of S-formulas and φ be an S-formula. We say that *ϕ is a consequence of Φ (written $\Phi \models \varphi$)* iff. for every S-interpretation $\mathcal{I}$ if $\mathcal{I} \models \psi$ for all $\psi \in \Phi$, then $\mathcal{I} \models \varphi$.

Definition 1.7. A formula φ is valid (written $\models \varphi$) iff. $\emptyset \models \varphi$.

Definition 1.8. A formula φ is *satisfiable* (written $Sat\varphi$) if and only if there is interpretation which is a model of φ. A set of formula Φ is *satisfiable* if and only if there is interpretation which is a model of Φ.

Lemma 1. *For all Φ and φ,*

$$\Phi \models \varphi \quad \textit{iff.} \quad \textit{not } Sat\Phi \cup \{\neg\varphi\}$$

Definition 1.9. Two interpretation $\mathcal{I}_1$ and $\mathcal{I}_2$ agree on $k \in S$ on x if $k^{\mathcal{A}_1} = k^{\mathcal{A}_2}$ or $\beta_1(x) = \beta_2(x)$.

Lemma 2. *Let's $\mathcal{I}_1 = (\mathcal{A}_1, \beta)$ be an S_1-interpretation and $\mathcal{I}_2 = (\mathcal{A}_2, \beta)$ be an S_2-interpretation. both with the same domain $A_1 = A_2$. Put $S := S_1 \cap S_2$.*

- *Let t be an S-term. If $\mathcal{I}_1$ and $\mathcal{I}_2$ agree on the S-symbols occuring in t and on the variables occuring in t, then $\mathcal{I}_1(t) = \mathcal{I}_2(t)$.*
- *Let φ be an S-formula. If $\mathcal{I}_1$ and $\mathcal{I}_2$ agree on the S-symbols and the variables occuring free in φ, then ($\mathcal{I}_1 \models \varphi$ iff. $\mathcal{I}_2 \models \varphi$).*

1.4 Lemmas in Satisfication Relation

Definition 1.10. Let $\mathcal{A}$ and $\mathcal{B}$ be S-Structures

- A map $\pi : A \to B$ is called an *isomorphism* of $\mathcal{A}$ onto $\mathcal{B}$ ($\pi : \mathcal{A} \simeq \mathcal{B}$) iff
  - π is a bijection of A onto B.
  - For n-ary $R \in S$ and $a_1, \ldots, a_n \in A$,

    $$R^{\mathcal{A}}(a_1, \ldots, a_n) \quad \text{iff} \quad R^{\mathcal{B}}(\pi(a_1), \ldots, \pi(a_n))$$

  - For n-ary $f \in S$ and $a_1, \ldots, a_n \in A$,

    $$\pi(f^{\mathcal{A}}(a_1, \ldots, a_n)) = f^{\mathcal{B}}(\pi(a_1), \ldots, \pi(a_n))$$

  - For $c \in S$,

    $$pi(c^{\mathcal{A}}) = c^{\mathcal{B}}$$

- Structure $\mathcal{A}$ and $\mathcal{B}$ are said to be *isomorphic* ($\mathcal{A} \simeq \mathcal{B}$) iff. there is an isomorphism $\pi : \mathcal{A} \simeq \mathcal{B}$.

Lemma 3. *For isomorphic S-structures $\mathcal{A}$ and $\mathcal{B}$ and every S-sentence φ,*

$$\mathcal{A} \models \varphi \quad \textit{iff.} \quad \mathcal{B} \models \varphi$$

Corollary 1. *If $\pi : \mathcal{A} \simeq \mathcal{B}$, then for $\varphi \in L_n^S$ and $a_0, \ldots, a_{n-1} \in A$,*

$$\mathcal{A} \models \varphi[a_0, \ldots, a_{n-1}] \quad \textit{iff.} \quad \mathcal{B} \models \varphi[\pi(a_0), \ldots, \pi(a_{n-1})]$$

Definition 1.11. Let $\mathcal{A}$ and $\mathcal{B}$ be S-structures. Then $\mathcal{A}$ is called a *substructure* of $\mathcal{B}$ ($\mathcal{A} \subseteq \mathcal{B}$) iff.

- $A \subseteq B$
-
  - for n-ary $R \in S$, $R^{\mathcal{A}} = R^{\mathcal{B}} \cap A^n$
  - for n-ary $f \in S$, $f^{\mathcal{A}}$ is the restriction of $f^{\mathcal{B}}$ to A^n
  - for $c \in S$, $c^{\mathcal{A}} = c^{\mathcal{B}}$

Lemma 4. *Let $\mathcal{A}$ and $\mathcal{B}$ be S-structures with $\mathcal{A} \subseteq \mathcal{B}$ and let $\beta : \{v_n | n \in \mathbb{N}\} \to A$ be an assignment in $\mathcal{A}$. Then the following holds for every S-term t*

$$(\mathcal{A}, \beta)(t) = (\mathcal{B}, \beta)(t)$$

and for every quantifier-free S-formula φ:

$$(\mathcal{A}, \beta) \models \varphi \quad \textit{iff.} \quad (\mathcal{B}, \beta) \models \varphi$$

Definition 1.12. The formulas which are derivable by means of the following calculus are called *universal formulas.*

$$\frac{}{\varphi}\ \text{if}\ \varphi\ \text{is quantifier-free}$$

$$\frac{\varphi, \psi}{(\varphi \star \psi)}\ \text{for}\ \star \in \{\wedge, \vee\}$$

$$\frac{\varphi}{\forall x \varphi}$$

Lemma 5. *Let $\mathcal{A}$ and $\mathcal{B}$ be S-structures with $\mathcal{A} \subseteq \mathcal{B}$ and let $\varphi \in L_n^S$ be a universal. Then the following holds for all $a_0, \ldots, a_{n-1} \in A$:*

$$\textit{If}\, \mathcal{B} \models \varphi[a_0, \ldots, a_{n-1}],\ \textit{then}\, \mathcal{A} \models \varphi[a_0, \ldots, a_{n-1}]$$

Corollary 2. *If $\mathcal{A} \subseteq \mathcal{B}$, then the following holds for every universal sentence φ:*

$$\textit{If}\, \mathcal{B} \models \varphi,\ \textit{then}\, \mathcal{A} \models \varphi$$